https://www.linkedin.com/feed/update/urn:li:activity:6572703037478051840

Let (x_n) be a sequence defined by $x_1 = 2$ and $x_{n+1} = \sqrt{x_n+8} - \sqrt{x_n+3}$, $\forall n \in \mathbb{N}$.

a) Prove that (x_n) is convergent and find its limit.

b) For each positive integer n prove that

$$n \le x_1 + x_2 + \ldots + x_n \le n+1.$$

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a) Let $h(x) := \sqrt{x+8} - \sqrt{x+3} = \dfrac{5}{\sqrt{x+8}+\sqrt{x+3}}$.

Noting that $h(x)$ strictly decrease on $(0,\infty)$ (calculus don't needed because $\sqrt{x+8}+\sqrt{x+3}$ strictly increase on $(0,\infty)$) and $h(1) = 1$ we can conclude that $x = 1$ is the only solution of equation $h(x) = x$ on $(0,\infty)$.

We will prove that $\lim\limits_{n\to\infty} x_n = 1$. Note that $\left|x_{n+1} - 1\right| = \left|1 - \dfrac{5}{\sqrt{x_n+8}+\sqrt{x_n+3}}\right| \le$
$\dfrac{\left|\sqrt{x_n+8}-3\right| + \left|\sqrt{x_n+3}-2\right|}{\sqrt{x_n+8}+\sqrt{x_n+3}} \le \dfrac{|x_n-1|}{\sqrt{x_n+8}+\sqrt{x_n+3}}\left(\dfrac{1}{\sqrt{x_n+8}+3} + \dfrac{1}{\sqrt{x_n+3}+2}\right) <$
$|x_n-1| \cdot \dfrac{1}{\sqrt{8}+\sqrt{3}} \cdot \left(\dfrac{1}{\sqrt{8}+3} + \dfrac{1}{\sqrt{3}+2}\right)$. Since $\sqrt{8}+\sqrt{3} > 4$, $\sqrt{8}+3 > 5$ and $\sqrt{3}+2 > 3$ then $\dfrac{1}{\sqrt{8}+\sqrt{3}} \cdot \left(\dfrac{1}{\sqrt{8}+3} + \dfrac{1}{\sqrt{3}+2}\right) < \dfrac{1}{4}\left(\dfrac{1}{5}+\dfrac{1}{3}\right) = \dfrac{2}{15}$ and, therefore, $\left|x_{n+1} - 1\right| < \dfrac{2}{15} \cdot |x_n - 1|, n \in \mathbb{N} \Leftrightarrow |x_n - 1| < \left(\dfrac{2}{15}\right)^{n-1}|x_1 - 1| = \left(\dfrac{2}{15}\right)^{n-1}$.

Hence, $\lim\limits_{n\to\infty}(x_n - 1) = 0 \Leftrightarrow \lim\limits_{n\to\infty} x_n = 1$.

b) Let $S_n := x_1 + x_2 + \ldots + x_n, n \in \mathbb{N}$. We will prove using Math Induction that

(**1**) $\quad n \le S_n \le n+1$.

For $n = 1$ the inequality obviously holds. Also, since $x_2 = \sqrt{10} - \sqrt{5} < 1$ ($\sqrt{10} < \sqrt{5}+1 \Leftrightarrow 10 < 6 + 2\sqrt{5} \Leftrightarrow 2 < \sqrt{5}$) we have $2 < S_2 < 3$.

For the Step of Math Induction we have to do some preparations.

Let $h_2(x) := h(h(x))$ and $g(x) := x + h(x)$. Since $h(x)$ decrease on $(0,\infty)$ then $h_2(x)$ increase on $(0,\infty)$. Also $g(x)$ increase on $(0,\infty)$ as well. Indeed, for any $0 < x_1 < x_2$ we have $g(x_2) - g(x_1) = x_2 - x_1 + \sqrt{x_2+8} - \sqrt{x_1+8} - \left(\sqrt{x_2+3} - \sqrt{x_1+3}\right) =$
$x_2 - x_1 - \dfrac{x_2 - x_1}{\sqrt{x_2+3}+\sqrt{x_1+3}} + \dfrac{x_2 - x_1}{\sqrt{x_2+8}+\sqrt{x_1+8}} > \dfrac{(x_2 - x_1)\left(\sqrt{x_2+3}+\sqrt{x_1+3}-1\right)}{\sqrt{x_2+3}+\sqrt{x_1+3}} > 0$

Since $x_2 < 1$ and $x_1 > 1$ then for any $n \in \mathbb{N}$ assuming $x_{2n} < 1$ and $x_{2n-1} > 1$ we obtain $x_{2n+2} = h_2(x_{2n}) < h_2(1) = 1$ and $x_{2n+1} = h_2(x_{2n-1}) > h_2(1) = 1$

Thus, by Math Induction $x_{2n} < 1$ and $x_{2n-1} > 1$ for any $n \in \mathbb{N}$. Therefore, since $g(1) = 1 + \sqrt{1+8} - \sqrt{1+3} = 2$ we obtain $x_{2n} + x_{2n+1} = g(x_{2n}) < g(1) = 2$, $x_{2n-1} + x_{2n} = g(x_{2n-1}) > g(1) = 2$. And now we ready to complete the proof of (**1**).

For any $n \ge 2$ assuming $k \le S_k \le k+1$ for $k = n-1, n$ we have:

1. If $n = 2m$ then $S_{n+1} = S_{n-1} + x_{2m} + x_{2m+1} < n+2$ and $S_{n+1} = S_n + x_{2m+1} > n+1$;
2. If $n = 2m-1$ then $S_{n+1} = S_n + x_{2m} > n+1$ and $S_{n+1} = S_{n-1} + x_{2m-1} + x_{2m} < n+2$.

Thus, the proof is complete.